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Maslov form and J-volume of totally real immersions

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Abstract

On every totally real submanifold M^n of $\mathbb{C}^n$, one can define a Maslov class analogous to the one defined for the Lagrangian submanifolds of $\mathbb{C}^n$. We define here a closed 1-form, expressed in terms of the extrinsic local geometric invariants of M^n and the complex structure of $\mathbb{C}^n$, whose cohomology class is the Maslov class of M^n. This generalizes to the totally real case, the result of Morvan (1981). This 1-form can still be defined if the ambient space $\mathbb{C}^n$ is substituted by a Kähler manifold $\tilde{M}^{2n}$, but it is not closed in general. However, we can build a variational problem on the space of totally real immersions, whose critical points are totally real submanifolds whose form defined above vanishes identically. In the case where $\tilde{M}^{2n} = \mathbb{C}^n$, we give a characterization and many examples of such submanifolds. Finally we study the second variation and prove a stability result for the critical submanifolds of a Kähler manifold with non-positive Ricci tensor. This extends the well-known results on Lagrangian submanifolds of $\tilde{M}^{2n}$. © 1998 Elsevier Science B.V.

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1. Introduction

Introduced by Maslov [12] in 1965 to study the asymptotic behaviour of solutions of some partial differential equations, the Maslov class proved to be a natural object in many fields of physics (especially in optics cf. [2,9]). This class is defined on every Lagrangian submanifold of the cotangent bundle of a manifold (and more generally on the base of every symplectic bundle provided with two Lagrangian sub-bundles). In particular, it is a symplectic invariant for Lagrangian submanifolds of $\mathbb{C}^n$: Two Lagrangian immersions

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j_0, j_1 of a manifold M^n in $\mathbb{C}^n$, which have different Maslov classes, are not homotopic through the Lagrangian immersions.

Arnold has obtained the Maslov class as the pull back, via the Gauss map, of a generator of the first cohomology group of the Lagrangian Grassmannian $\Lambda(n)$ (cf. [1], for instance). In 1981, Morvan [13] gave a Riemannian interpretation of this class for Lagragian submanifolds of $\mathbb{C}^n$. He defined a closed differential 1-form μ on the submanifold expressed in terms of the mean curvature vector H and the canonical symplectic structure Ω of $\mathbb{C}^n$: $\mu = (n/\pi) i_H \Omega$ (i is the inner product). Its cohomology class is the Maslov class.

In the first part of this article, we extend the preceding results to the case of totally real submanifolds of $\mathbb{C}^n$. We define on every totally real immersion $j : M^n \longrightarrow \mathbb{C}^n$ a differential 1-form $\mu(j)$ (the *Maslov form* of j) by taking the pull back, by the Gauss map, of a canonical 1-form living in the totally real Grassmannian. This 1-form induces an integer cohomology class $[\mu(j)]$ which we call the *Maslov class* of the immersion j. This class allows us to detect homotopy classes of totally real immersions. (If j_0 and j_1 are two totally real immersions of M^n and if $[\mu(j_0)] \neq [\mu(j_1)]$, then j_0 and j_1 are not homotopic through totally real immersions). We give a Riemannian representative of this class in terms of the local geometry of the submanifold.

Theorem 1. *The Maslov form of a totally real immersion $j : M^n \longrightarrow \mathbb{C}^n$ is the 1-form:*

$$\mu(X) = \frac{1}{\pi}\mathrm{Tr}(F^{-1} h_X) \qquad \forall X \in TM^n,$$

where $h_X = h(X, \cdot) : TM^n \longrightarrow T^\perp M^n$ is the contraction of the second fundamental form h with X and $F : TM^n \longrightarrow T^\perp M^n$ the projection over the normal bundle of the complex structure J : $FX = (JX)^\perp$.

As in the Lagrangian case, we can define a Maslov form on every totally real immersion of a manifold M^n in a Kähler manifold $\tilde{M}^{2n}$. This one is closed as soon as $\tilde{M}^{2n}$ has a first Chern form γ_1 which vanishes identically.

In the Lagrangian case, the formula of Morvan brings out a deep relationship between minimality of a Lagrangian submanifold and the Maslov form. One has $H \equiv 0$ if and only if $\mu \equiv 0$. In the more general context of isotropic submanifolds, Chen and Morvan [6] have also brought to the fore the relationship between Maslov form and a variation problem. It is tempting to try to obtain a similar result in the totally real case, but theorem 1 prevents us from a so simple relation. We are thus induced to consider a new variational problem on the space of totally real immersions such that critical points are immersions with zero Maslov form. We define a volume form vol_J which takes the complex structure J of $\tilde{M}^{2n}$ into account: the J-volume. If $\Phi : M^n \times]-\epsilon, \epsilon[\longrightarrow \tilde{M}^{2n}$ is a deformation of M^n (i.e. a one-parameter family of totally real immersions such that $\Phi_0 = j$) and if $v(t)$ denotes the J-volume of $\Phi_t(M^n)$, then we obtain:

Proposition 2. *Let $j : M^n \longrightarrow \tilde{M}^{2n}$ be a totally real immersion. Then*

$$v'(0) = \pi \int_{M^n} \mu(X)\,\mathrm{vol}_J + \int_{\partial M^n} i_T(\mathrm{vol}_J).$$

We have written $(\partial\Phi/\partial t) = JX + T \in JTM^n \oplus TM^n$ the deformation field. We deduce the following:

Theorem 2. *Let $j : M^n \longrightarrow \tilde{M}^{2n}$ be a totally real immersion. Then j is critical for the J-volume V_J (among totally real immersions that leave the boundary fixed) if and only if its Maslov form vanishes identically.*

In Section 4, we give a geometric characterization of totally real immersions in $\mathbb{C}^n$ which have a vanishing Maslov form (STR immersions). This allows us to find numerous examples of such immersions.

We study next the stability problem. It is well known that minimal Lagrangian submanifolds of $\mathbb{C}^n$ are stable. This result can be achieved using a specific calibration of $\mathbb{C}^n$ [10]. In order to study stability of totally real immersions, we broaden lightly the notion of calibration. We define the J-calibrations and obtain the following result.

Theorem 5. *Every totally real submanifold M^n of $\mathbb{C}^n$, which is critical for the J-volume, minimizes homologically the J-volume, i.e. $V_J(M^n) \leq V_J(N^n)$ for every submanifold N^n such that $\partial N^n = \partial M^n$ and $[M^n - N^n] = 0$ in $H_n(\tilde{M}^{2n})$.*

In the general case ($\tilde{M}^{2n}$ any Kähler manifold), the computation of the second variation of the J-volume leads to the following stability result (compare with [5,15] for the Lagrangian case and [6] for the isotropic case):

Theorem 6. *Let $\tilde{M}^{2n}$ be a Kähler manifold with non-positive Ricci curvature, then every immersion $j : M^n \longrightarrow \tilde{M}^{2n}$ critical for the J-volume is stable for the J-volume.*

2. Maslov form and Maslov class

2.1. Geometry of $\mathcal{T}(n)$ – The Berger fibration [14]

The Grassmannian $\mathcal{T}(n)$ of totally real n-planes of $\mathbb{C}^n$ can be identified with the symmetric space $Gl(n,\mathbb{C})/Gl(n,\mathbb{R})$. As any symmetric space, $\mathcal{T}(n) \simeq Gl(n,\mathbb{C})/Gl(n,\mathbb{R})$ is a fibre bundle over a compact symmetric space, which is here $U(n)/O(n)$. The fibre is a vector space which can be identified with the space of real antisymmetric matrices. The map that realizes this fibration is called the Berger fibration. It is easily described by means of the Mostov decomposition: Every matrix M of $Gl(n,\mathbb{C})$ can be written $M = U\mathrm{e}^{\mathrm{i}A}\mathrm{e}^{S}$, where U is unitary, A real antisymmetric and S real symmetric. The Berger fibration is the map defined by

$$\begin{aligned} \phi : Gl(n,\mathbb{C})/Gl(n,\mathbb{R}) &\longrightarrow U(n)/O(n) \\ [M = U\mathrm{e}^{\mathrm{i}A}\mathrm{e}^{S}] &\longmapsto [U]. \end{aligned}$$

The space $U(n)/O(n)$ (which can be identified with the Lagrangian Grassmannian) is again a fibre bundle over $\mathbb{S}^1$ via the map $\det^2$. One has the following diagram:

$$Gl(n,\mathbb{C})/Gl(n,\mathbb{R}) \xrightarrow{\phi} U(n)/O(n) \xrightarrow{det^2} \mathbb{S}^1,$$

where each arrow is a fibration. Let $\omega = (1/2i\pi)(dz/z)$ be the volume form of $\mathbb{S}^1$. Pulling back ω, one gets a closed 1-form α on $Gl(n,\mathbb{C})/Gl(n,\mathbb{R})$. The value of α at the identity is given by

$$\alpha_{I\mathrm{d}}(T) = \frac{1}{\pi}\mathrm{Tr}(\mathrm{Im}\, T) \quad \forall T \in gl(n,\mathbb{C})/gl(n,\mathbb{R}) = \mathcal{M}_n(\mathbb{C})/\mathcal{M}_n(\mathbb{R}).$$

Definition. The form α (resp. the integer class $[\alpha]$) is called the *Maslov form* (resp. the *Maslov class*) of $\mathcal{T}(n)$.

2.2. *Maslov form of a totally real immersion in* $\mathbb{C}^n$

In what follows, $\langle\ ,\ \rangle$ denotes the usual scalar product of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, J the complex structure and $\Omega = \langle\cdot, J\cdot\rangle$ the symplectic form. Let M^n be a n-dimensional manifold and $j : M^n \longrightarrow \mathbb{C}^n$ any immersion. We put the induced metric $j^*(\langle\ ,\ \rangle)$ on M^n and thus j becomes an isometric immersion. We shall often write $\langle\ ,\ \rangle$ instead of $j^*(\langle\ ,\ \rangle)$ and M^n instead of $j(M^n)$ if the context is clear. At each point p of M^n, we denote by T_pM^n the tangent space and $T_p^{\perp}M^n$ the normal space. An immersion j is called *totally real* if for every point p one has $T_pM^n \oplus JT_pM^n = \mathbb{C}^n$. If, moreover $JT_pM^n = T_p^{\perp}M^n$, the immersion is called *Lagrangian*. The second fundamental form of j is denoted by $h : TM^n \times TM^n \longrightarrow T^{\perp}M^n$. It is well known that h is a symmetric tensor given by

$$h(X,Y) = \tilde{\nabla}_XY - \nabla_XY \quad \forall X, Y \in TM^n,$$

where $\tilde{\nabla}$ is the canonical connection of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and ∇ the Levi-Civita connection of M^n. One puts $H = (1/n)\mathrm{Tr}\ h$. This defines a normal vector field called the *mean curvature vector field*. If H vanishes identically, j is said to be minimal.

Let $j : M^n \longrightarrow \mathbb{C}^n$ be a totally real immersion.

Definition. The (closed) 1-form defined by $\mu(j) = (1/\pi)G^*\alpha$ is called the *Maslov form* of the totally real immersion j. (The Gauss map G of j takes its values in $\mathcal{T}(n)$). The *Maslov class* of j is the cohomology class $[\mu(j)]$.

The Maslov class is (obviously) an invariant for totally real immersions. In particular, if j_0 and j_1 are two totally real immersions and if $[\mu(j_0)] \neq [\mu(j_1)]$, then j_0 and j_1 are not homotopic through totally real immersions.

We shall often write μ instead of $\mu(j)$.

Theorem 1. *The Maslov form of a totally real immersion* $j : M^n \longrightarrow \mathbb{C}^n$ *is given by the 1-form:*

$$\mu(X) = \frac{1}{\pi}\mathrm{Tr}(F^{-1}h_X) \qquad \forall X \in TM^n,$$

where $h_X = h(X,\cdot) : TM^n \longrightarrow T^{\perp}M^n$ *is the contraction of the second fundamental form* h *with* X *and* $F : TM^n \longrightarrow T^{\perp}M^n$ *is the projection over the normal bundle of the complex structure* J : $FX = (JX)^{\perp}$.

Remark. This formula is a generalization of the formula of Morvan for the Lagrangian case [13].

Proof of Theorem 1. Let X be a vector field on M^n, one has

$$\mu(X) = \alpha(\mathrm{d}G(X)) = \frac{1}{\pi}\mathrm{Tr}\,\mathrm{Im}(\mathrm{d}G(X)).$$

Let $h_X^{\mathbb{C}} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the J-linear extension of $h(X,\cdot) : TM \longrightarrow T^{\perp}M$. One can write

$$\mu(X) = \frac{1}{\pi}\mathrm{Tr}\,\mathrm{Im}(h_X^{\mathbb{C}}).$$

Let p be a point of M^n. There always exists an orthonormal frame $(e_1,\ldots,e_n)$ of T_pM^n for which $\Omega_{|T_pM}$ can be put into the form

$$\Omega_{|T_pM} = \sum_{k=1}^{k=[n/2]} \lambda_k \theta^{2k-1}\wedge\theta^{2k}.$$

Here $(\theta^1,\ldots,\theta^n)$ denotes the dual frame and $\lambda_1,\ldots,\lambda_n$, n real numbers. The immersion j being totally real, $(e_1,\ldots,e_n, Je_1,\ldots,Je_n)$ is a frame of $\mathbb{C}^n$ (but it is not orthonormal in general). One completes $(e_1,\ldots,e_n)$ in an orthonormal frame of $\mathbb{C}^n$ by putting

$$\epsilon_1 = F(e_1)/\|F(e_1)\|,\ldots,\epsilon_n = F(e_n)/\|F(e_n)\|.$$

We denote by $(\theta^1,\ldots,\theta^n,\theta^{1*},\ldots,\theta^{n*})$ the dual forms of the frame $(e_1,\ldots,e_n, Je_1,\ldots,Je_n)$. One has

$$\theta^{k*} = \left\langle \frac{e_j}{\|F(e_j)\|}, e_k\right\rangle.$$

Thus,

$$\begin{aligned}
&\frac{1}{\pi}\mathrm{Tr}\,\mathrm{Im}(h_X^{\mathbb{C}})\\
&= \frac{1}{\pi}\sum_k \theta^{k*}(h(X,e_k)) = \frac{1}{\pi}\sum_k \theta^{k*}\left(\sum_j h^j(X,e_k)\epsilon_j\right)\\
&= \frac{1}{\pi}\sum_k\left\langle\sum_j h^j(X,e_k)\frac{e_j}{\|F(e_j)\|}, e_k\right\rangle = \frac{1}{\pi}\sum_k\left\langle\sum_j h^j(X,e_k)F^{-1}(\epsilon_j), e_k\right\rangle\\
&= \frac{1}{\pi}\sum_k\langle F^{-1}h(X,e_k), e_k\rangle = \frac{1}{\pi}\mathrm{Tr}(F^{-1}h_X).
\end{aligned}$$

This finishes the proof of Theorem 1. □

2.3. Particular cases and applications

In the following examples, we compute explicitly the Maslov form of some particular totally real immersions. Under certain conditions, the Maslov form can be expressed in terms of the mean curvature vector and the Lichnerowicz–Wirtinger angle.

Definition. For every vector $X \in T_pM$, the angle $0 \leq \theta(p, X) \leq \frac{1}{2}\pi$ between JX and T_pM is called the *Lichnerowicz–Wirtinger angle* of X.

2.3.1. Totally real immersions such that $\theta(p, X)$ is independent of X

Let $j : M^n \longrightarrow \mathbb{C}^n$ be a totally real immersion such that the Lichnerowicz–Wirtinger angle is independent of X. Let $(e_1, \ldots, e_n)$ denote an orthonormal frame such that

$$\Omega_{|T_pM} = \sum_{k=1}^{k=[n/2]} \lambda_k \theta^{2k-1} \wedge \theta^{2k},$$

where $(\theta^1, \ldots, \theta^n)$ is dual to $(e_1, \ldots, e_n)$ and $\lambda_k \in \mathbb{R}$. As the angle $\theta(p, X)$ does not depend on X, one defines in a natural way, a function $\theta : M^n \longrightarrow [0, \frac{1}{2}\pi]$. One has:

(1) if n is even, $\Omega_{|T_pM^n} = \cos\theta(p) \sum_{k=1}^{k=n/2} \theta^{2k-1} \wedge \theta^{2k}$,

(2) if n is odd, $\Omega_{|T_pM^n} \equiv 0$.

Let $P : TM^n \longrightarrow TM^n$ denote the orthogonal projection of JX on TM^n. One has $\Omega_{|T_pM}(X, Y) = \langle PX, Y \rangle$. In a Lagrangian point, P is zero. We denote by $\mathrm{Lag}(j, M^n)$ the set of Lagrangian points of the immersion j. One can define on $M^n \setminus \mathrm{Lag}(j, M^n)$ an almost complex structure i by

$$i = \frac{1}{\cos\theta} P.$$

In a Lagrangian point, we put $i = 0$.

Proposition 1. *Let $j : M^n \longrightarrow \mathbb{C}^n$ be a totally real immersion such that its Lichnerowicz–Wirtinger angle does not depend on X. Then the Maslov form of j is given by*

$$\mu(X) = \frac{1}{\pi \sin^2\theta}(-n\langle JH, X \rangle + \sin\theta \, \mathrm{d}\theta \circ i(X)).$$

Proof. If the dimension of M^n is odd, then j is Lagrangian and the Maslov form is given by $\mu(X) = (n/\pi)\langle JX, H \rangle$. If the dimension is even, one can write

$$\mu(X) = \frac{1}{\pi}\mathrm{Tr}(F^{-1}h_X) = \frac{1}{\pi \sin\theta}\mathrm{Tr}(h_X).$$

Deriving the scalar products $\langle Je_i, e_j \rangle$, one gets:

(1) if $(i, j) = (2k-1, 2k)$ or $(2k, 2k-1)$,

$$h^{2k-1}_{2k-1,2k} - h^{2k}_{2k-1,2k-1} = -\mathrm{d}\theta(e_{2k-1}), \quad h^{2k}_{2k,2k-1} - h^{2k-1}_{2k,2k} = \mathrm{d}\theta(e_{2k}),$$

(2) for the other couples (i, j), $h^i_{ij} - h^j_{ii} = 0$.
Thus,

$$\begin{aligned}\mathrm{Tr}(h_X) &= \sum_{i,j} h^i_{ij} X^j \\ &= \frac{n}{\sin\theta}\langle H, JX\rangle + \mathrm{d}\theta\left(\sum_{k=1}^{p}(X^{2k-1}e_{2k} - X^{2k}e_{2k-1})\right).\end{aligned}$$

But $i(e_{2k-1}) = e_{2k}$ and $i(e_{2k}) = -e_{2k-1}$, therefore

$$\mu(X) = \frac{1}{\pi\sin^2\theta}(n\langle H, JX\rangle + \sin\theta\,\mathrm{d}\theta \circ i(X)). \qquad \square$$

Here are some applications.

Two-dimensional submanifolds. The Lichnerowicz–Wirtinger angle is always independent of X. Proposition 1 gives us the expression of the Maslov form for every totally real immersion $j : M^2 \longrightarrow \mathbb{C}^2$. We then obtain the same expression as in [7].

Cayley immersions. Let $\mathbb{O}$ denote the Cayley algebra and let $j : M^4 \longrightarrow \mathbb{O}$ be an immersion. As a vector space, $\mathbb{O}$ is spanned by $1, i, j, k, e, ei, ej, ek$ where $1, i, j, k$ span the quaternionic ring $\mathbb{H}$. One has $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$. Multiplication by e defines a complex structure J_e on $\mathbb{O}$. Let $\Omega = \langle J_e\cdot, \cdot\rangle$ be the associated Kähler form and define a closed 4-form by

$$\Phi = -\tfrac{1}{2}\Omega \wedge \Omega + \mathrm{Re}(\mathrm{d}z).$$

Definition. Any immersion $j : M^4 \longrightarrow \mathbb{O}$ such that $\Phi(T_pM^4) = 1$ for all p in M^4 is called a *Cayley immersion*.

Remark. In fact, Φ is a calibration and Cayley submanifolds are Φ-submanifolds for this calibration. In particular, they are minimal [10].

For every Cayley immersion, $\Omega_{|T_pM^4}$ has the following expression [10]:

$$\Omega_{|T_pM^4} = \cos(\theta(p))(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4),$$

where $(\theta^1, \ldots, \theta^4)$ is the dual basis of an orthonormal basis $(e_1, \ldots, e_4)$ of T_pM^4 and where $\theta(p) \in [0, \frac{1}{2}\pi]$. If $\theta = \frac{1}{2}\pi$ then M^4 is Lagrangian and if $\theta \equiv 0$, M^4 is complex or anticomplex. Thus, let us assume $\theta(p) \in]0, \frac{1}{2}\pi]$ for all point p of M^4, i.e. M^4 is totally real. Using Proposition 1, we can write

$$\mu(X) = \frac{1}{\pi\sin^2\theta}(4\langle H, JX\rangle + \sin\theta\, d\theta \circ i(X)).$$

Any Cayley submanifold is minimal, thus one has

$$\mu(X) = \frac{\mathrm{d}\theta \circ i(X)}{\pi\sin\theta}.$$

Explicit examples of Cayley submanifolds are given in [10].

Slant submanifolds. An immersion $j : M^n \longrightarrow \mathbb{C}^n$ is called *slant* if for all point $p \in M^n$ and all $X \in TM^n$, the angle $\theta_p(X) = \angle(JX, T_pM^n)$ is constant. In that case, if $\theta \equiv \frac{1}{2}\pi$ then j is a Lagrangian immersion and if $\theta = 0$, j is holomorphic or antiholomorphic. We assume now $\theta \neq \frac{1}{2}\pi$ and $\theta \neq 0$. Under these hypothesis, the manifold M^n has to be of even dimension. Proposition 1 gives us

$$\mu(X) = \frac{1}{\pi \sin^2\theta}\langle H, JX\rangle.$$

We find here a well-known formula [5].

2.3.2. An explicit example of torus $\mathbb{T}^2$ embedded in $\mathbb{C}^2$ with zero Maslov class

In 1990, Viterbo [17] has shown that every Lagrangian embedded torus $\mathbb{T}^n$ in $\mathbb{C}^n$ has a non-zero Maslov class. The situation is quite different if one only assumes that the torus is totally real. Fiedler [8] and Polterovich [16] have built, by topological methods, examples of totally real tori with zero Maslov class. We give here an explicit new example of such a torus.

Let j_λ be the family of embeddings of $\mathbb{T}^2$ defined by

$$j_\lambda : \mathbb{T}^2 \longrightarrow \mathbb{S}^3 \subset \mathbb{C}^2$$
$$(u, v) \longmapsto \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \frac{1}{2}\left\{ f(u, v)\begin{pmatrix} \cos u \\ \cos v \\ \sin u \\ \sin v \end{pmatrix} + g(u, v)\begin{pmatrix} \sin v \\ -\sin u \\ \cos v \\ -\cos u \end{pmatrix}\right\}$$

with

$$f(u, v) = +\sqrt{2}\sin(\varphi(u, v) - \tfrac{1}{4}\pi),$$
$$g(u, v) = -\sqrt{2}\cos(\varphi(u, v) - \tfrac{1}{4}\pi),$$
$$\varphi(u, v) = \lambda\cos(u + v);\ \ \lambda \in \mathbb{R}_+^*.$$

It is easily checked that:

(1) The embedding j_λ is totally real for the complex structure J given by $J(x, y, z, t) = (-z, -t, x, y)$.

(2) The embedding j_λ is Lagrangian for an other complex structure J_1 given by $J_1(x, y, z, t) = (-t, -z, y, x)$ and therefore its Maslov class for this other structure is non-zero.

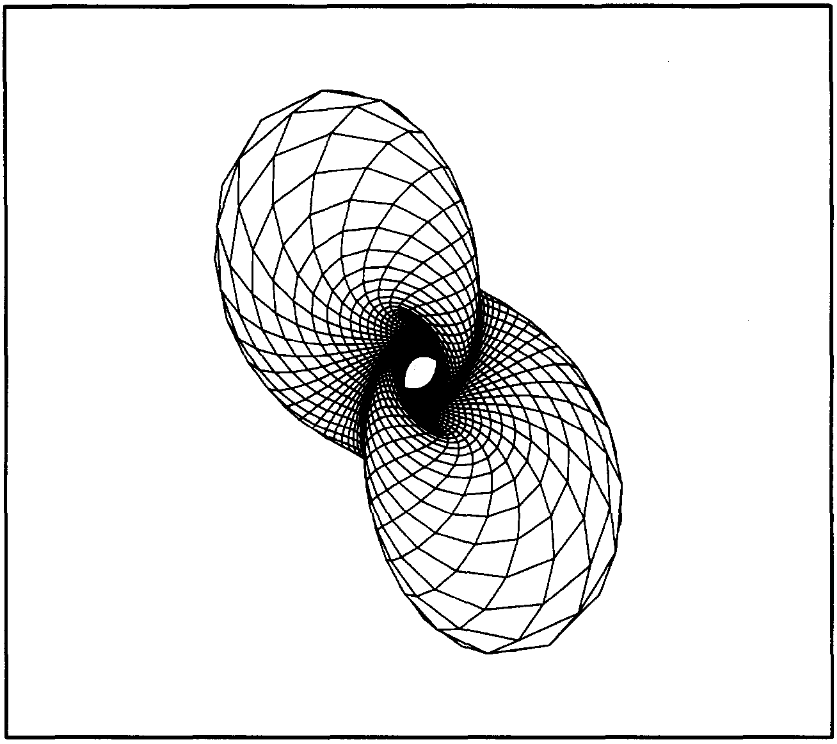
(3) The image $j_\lambda(\mathbb{T}^2)$ is a torus in $\mathbb{S}^3 \subset \mathbb{C}^2$. Precisely, $j_\lambda(\mathbb{T}^2)$ is a Hopf torus, i.e. the reciprocal image of a closed curve in $\mathbb{S}^2$ by the Hopf fibration. Thus, this torus is flat.

The Maslov form (for the structure J) of j_λ can be performed using Theorem 1. One gets

$$\mu(j_\lambda) = \frac{1}{\pi}\frac{\cos(2\varphi(u, v))\sin(u + v)(1 + \lambda^2\sin^4(u + v))}{\delta(u, v)}(\mathrm{d}u - \mathrm{d}v),$$

with

$$\delta(u, v) = 1 - \cos^2(2\varphi(u, v))\cos^2(u + v) + 4\cos^2(2\varphi(u, v))\sin^4(u + v)\lambda^2$$
$$-2\sin(4\varphi(u, v))\sin^2(u + v)\cos(u + v)\lambda.$$



Scheme 1.

We put

$$G(u, v) = \frac{\cos(2\varphi(u, v))\sin(u + v)(1 + \lambda^2 \sin^4(u + v))}{\delta(u, v)}.$$

Functions $u \longmapsto G(u, v_0)$ and $v \longmapsto G(u_0, v)$ are odd, so $[\mu(j_\lambda)] = 0$. Scheme 1 is the stereographic projection of $j_{\pi/4}(\mathbb{T}^2)$.

2.4. Extension to Kähler case and relationship with the first Chern class

Until now, we have assumed that the ambient manifold was the complex space $\mathbb{C}^n$. We wish to generalize these results to the case of a Kähler manifold $\tilde{M}^{2n}$. Let $j : M^n \longrightarrow \tilde{M}^{2n}$ be an (isometric) totally real immersion of M^n in $\tilde{M}^{2n}$, one can define the Maslov form μ of j by the formula

$$\mu(X) = \frac{1}{\pi}\mathrm{Tr}(F^{-1}h_X) \quad \forall X \in TM^n.$$

In general, this 1-form is not closed. Let $\tilde{\Omega}^{\mathbb{C}}$ be the curvature matrix seen as a complex matrix and let $\gamma_1 = (1/2i\pi)\mathrm{Tr}(\tilde{\Omega}^{\mathbb{C}})$ be the first Chern form of $\tilde{M}^{2n}$. Then one has $\mathrm{d}\mu = 2\gamma_1$. If $\gamma_1|_{j(M^n)} \equiv 0$ (in particular if γ_1 vanishes identically), the Maslov form is closed and one can define a Maslov class. In this context, it appears as a secondary characteristic class. (One could broaden again these results to the case of a complex bundle provided with two totally real sub-bundles).

3. The J-volume

We assume here that M^n is orientable, compact, eventually with boundary and that $\tilde{M}^{2n}$ is a Kähler manifold.

The vanishing of the Lagrangian Maslov form is a minimality condition for the volume functional in the space $\mathcal{L}(M^n, \tilde{M}^{2n})$ of Lagrangian immersions. Indeed, if $v(t)$ is a variation of the volume in the normal direction ξ of a Lagrangian submanifold L^n, one can write

$$v'(0) = n \int_{L^n} \langle H, \xi \rangle \operatorname{vol}_{L^n} = n \int_{L^n} \langle H, JX \rangle \operatorname{vol}_{L^n} = \pi \int_{L^n} \mu(X) \operatorname{vol}_{L^n},$$

where $X = -J\xi \in TL^n$ (we have assumed $\xi_{|\partial L^n} = 0$). Thus, L^n is minimal if and only if μ is identically zero (cf. [13]). We generalize these results to the space $\mathcal{TR}(M^n, \tilde{M}^{2n})$ of totally real immersions. The idea is to take the complex structure J into account. This leads us to define a new volume form: the J-volume.

3.1. Definition of the J-volume

Let j be a totally real immersion of M^n into $\tilde{M}^{2n}$, p a point of M^n and $(e_1, \ldots, e_n)$ a direct orthonormal frame of T_pM^n.

Definition.

(1) The *J-density* of M^n at a point p is the real number $\rho_J(p)$ defined by

$$\rho_J(p) = \sqrt{\Omega^n(e_1, \ldots, e_n, Je_1, \ldots, Je_n)}.$$

(2) The *J-volume* of M^n is the real number $V_J(j, M^n)$ defined by

$$V_J(j, M^n) = \int_{M^n} \rho_J \operatorname{vol}_{M^n}.$$

The real number $\rho_J(p)$ does not depend on the (direct) orthonormal frame and the function ρ_J is C^∞ (because it never vanishes). If M^n has a finite volume then the J-volume of M^n is finite because $0 < \rho_J \leq 1$. Moreover, if j is a Lagrangian immersion then $\rho_J \equiv 1$ and $V_J(j, M^n) = \operatorname{Vol}(M^n)$ in the usual sense.

3.2. First variation of the J-volume

Let j be an immersion of M^n in $\tilde{M}^{2n}$. A *deformation* Φ of j is C^∞ map from $M^n \times]-\epsilon, \epsilon[$ to $\tilde{M}^{2n}$ such that $\Phi_t = \Phi(\cdot, t)$ is a one-parameter family of immersions and such that

$\Phi_0 = j$. We put $M_t^n = \Phi_t(M^n)$ and $v(t) = V_J(j, M_t^n) = \int_{M_t^n} \rho_J(t)\mathrm{vol}_{M_t^n}$. If Φ is a deformation of a totally real immersion j then, for sufficiently small t, Φ_t is still a totally real immersion. Moreover, if the J-volume of $j(M^n)$ is finite, it will be the same for $M_t^n = \Phi_t(M^n)$. Therefore, the function $v(t) = V_J(j, M_t^n)$ is well-defined and C^∞.

Theorem 2. *Let $j : M^n \longrightarrow \tilde{M}^{2n}$ be a totally real immersion. Then j is critical for the J-volume V_J (among totally real immersions leaving the boundary fixed) if and only if its Maslov form vanishes identically.*

This theorem is a straightforward consequence of:

Proposition 2. *Let $j : M^n \longrightarrow \tilde{M}^{2n}$ be a totally real immersion. Then*

$$v'(0) = \pi \int_{M^n} \mu(X)\rho_J \,\mathrm{vol}_{M^n} + \int_{\partial M^n} i_T(\rho_J \,\mathrm{vol}_{M^n}).$$

Proof. It is easy to check that

$$\rho_J(t)\Phi^*(\mathrm{vol}_{M_t^n}) = \sqrt{\|\mathrm{d}\tilde{\Phi}_t(e_1) \wedge \cdots \wedge \mathrm{d}\tilde{\Phi}_t(Je_n)\|}\,\mathrm{vol}_{M^n},$$

where $\mathrm{d}\tilde{\Phi}$ is the J-linear extension of $\mathrm{d}\Phi$. We put $\mathrm{d}\tilde{\Phi}_t(e_i) = \tilde{e}_i$ (if $t = 0$ then $\tilde{e}_i = e_i$) and $G(t) = \|\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n \wedge J\tilde{e}_1 \wedge \cdots \wedge J\tilde{e}_n\|^2$. Thus,

$$v(t) = V_J(j, M_t^n) = \int_{M^n} G^{1/4}(t)\,\mathrm{vol}_{M^n}.$$

One has

$$\frac{\mathrm{d}G(t)}{\mathrm{d}t}|_{t=0} = 2\sum_i \langle \tilde{e}_1 \wedge \cdots \wedge \tilde{\nabla}_{\partial/\partial t}\tilde{e}_i \wedge \cdots \wedge J\tilde{e}_n, \tilde{e}_1 \wedge \cdots \wedge J\tilde{e}_n\rangle|_{t=0}$$
$$+2\sum_i \langle \tilde{e}_1 \wedge \cdots \wedge \tilde{\nabla}_{\partial/\partial t}J\tilde{e}_i \wedge \cdots \wedge J\tilde{e}_n, \tilde{e}_1 \wedge \cdots \wedge J\tilde{e}_n\rangle|_{t=0}.$$

The flow of E is Φ_t and $\tilde{e}_i = (\Phi_t)_* e_i$, so $[e_i, E] = 0$. Thus,

$$\left.\frac{\mathrm{d}G(t)}{\mathrm{d}t}\right|_{t=0} = 2G(0)\sum_i (\theta^i(\tilde{\nabla}_{e_i}E) + \theta^{i*}(J\tilde{\nabla}_{e_i}E))$$
$$= 4G(0)\left\{-\sum_i \theta^{i*}(\tilde{\nabla}_{e_i}X) + \sum_i \theta^i(h(e_i, T)) + \mathrm{div}(T)\right\}.$$

But $(1/\pi)\sum_i \theta^{i*}(\tilde{\nabla}_{e_i}X) = \mu(X)$ and a direct computation gives $\sum_i \theta^i(h(e_i, T)) = \mathrm{d}\ln\rho_J(T)$ for all T in TM^n. Therefore,

$$v'(0) = \int_{M^n} \{-\pi\,\mu(X) + \mathrm{d}\ln\rho_J(T) + \mathrm{div}(T)\}\,G(0)^{1/4}\,\mathrm{vol}_{M^n}.$$

It ensues from $\{\mathrm{d}\ln\rho_J(T)+\mathrm{div}(T)\}\rho_J\,\mathrm{vol}_{M^n}=\mathrm{d}(i_T(\rho_J\,\mathrm{vol}_{M^n}))$ and from $G(0)=\rho_J^4$ that

$$v'(0)=-\pi\int_{M^n}\mu(X)\rho_J\,\mathrm{vol}_{M^n}+\int_{\partial M^n}i_T(\rho_J\,\mathrm{vol}_{M^n}).$$

This proves Proposition 2. □

4. Immersions in $\mathbb{C}^n$ which are critical for the J-volume

In the case where $\tilde{M}^{2n}=\mathbb{C}^n$, we give a geometric characterization of critical immersions for the J-volume.

4.1. Special totally real immersions (STR)

Let $(\epsilon_1,\ldots,\epsilon_n)$ be a (direct) orthonormal frame of $\mathrm{Re}(\mathbb{C}^n)$ and $j: M^n\to\mathbb{C}^n$ be any immersion. At a point p of M^n we denote by $(e_1(p),\ldots,e_n(p))$ a (direct) orthonormal frame of T_pM^n. Regarding $e_1(p),\ldots,e_n(p)$ as complex vectors, one can define a complex number $z(p)$ by the formula

$$e_1(p)\wedge\cdots\wedge e_n(p)=z(p)\epsilon_1\wedge\cdots\wedge\epsilon_n.$$

(Here $\wedge$ denotes the exterior product for the complex structure of $\mathbb{C}^n$). One has

$$\rho_J(p)=\sqrt{|\det(u_1,Ju_1,\ldots,u_n,Ju_n)|}=|\det(u_1^{\mathbb{C}},\ldots,u_n^{\mathbb{C}})|=|z(p)|.$$

An immersion j is totally real if and only if $|z(p)|>0$ for every point p of M^n. In that case, one writes $z(p)$ in polar coordinates $z(p)=r(p)\mathrm{e}^{\mathrm{i}\beta(p)}$.

Definition. Let $j: M^n\longrightarrow\mathbb{C}^n$ be an immersion and U be the open set on which j is totally real.
(1) If $\forall p\in U:\beta(p)=\beta=$ constant, j is said to be *special*.
(2) If j is special and $U=M^n$, j is said to be *special totally real (STR)*.

Remark. This definition broadens the one of Harvey and Lawson for special Lagrangian submanifolds [10].

Theorem 3. *Let $j: M^n\longrightarrow\mathbb{C}^n$ be a totally real immersion and $\mu(j)$ its Maslov form, then $\mu(j)\equiv 0$ if and only if j is STR.*

Proof. The Maslov form is obtained by means of the Berger fibration π, taking the pull-back of the volume form of $\mathbb{S}^1$ according to the diagram:

$$M^n\xrightarrow{G}Gl(n,\mathbb{C})/Gl(n,\mathbb{R})\xrightarrow{\pi}U(n)/O(n)\xrightarrow{\det^2}\mathbb{S}^1.$$

Since $\det \circ \pi = \det/|\det|$, $\mu \equiv 0$ if and only if $\det/|\det| =$ constant. The Jacobian of $\mathrm{d}j^{\mathbb{C}}$ is given by $\mathrm{Jac}_{\mathbb{C}}(\mathrm{d}j^{\mathbb{C}})(p) = r(p)\mathrm{e}^{\mathrm{i}\beta(p)}$. Therefore, the condition $\mu \equiv 0$ is equivalent to $\beta(p) = \beta =$ constant. □

4.2. *Examples of STR immersions*

There is no STR immersion of compact manifold (without boundary) in $\mathbb{C}^n$. In fact, if $j : M^n \longrightarrow \mathbb{C}^n$ was such an immersion then $(t+1)j$ would be an STR immersion for every $t > 0$ and thus one could define an STR deformation of j by putting $\Phi_t = (t+1)j$. Let $v(t) = V_J(\Phi_t(j), M^n)$. One would have

$$v(t) = V_J((t+1)j, M^n) = (t+1)^n V_J(j, M^n).$$

So $v'(t) = n(t+1)^{n-1} V_J(j, M^n)$ and therefore $v'(0) \neq 0$.

We give here some examples of STR immersions of (non-compact) manifolds in $\mathbb{C}^n$. Any restriction of such immersions to a compact domain is critical for the J-volume.

4.2.1. *STR immersions of an open set of the plane*

Let U be an open set of the plane and $\lambda : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^∞ function. An easy computation shows that

$$\begin{array}{cccc} j_\lambda : & U & \longrightarrow & \mathbb{C}^2 \\ & (u, v) & \longmapsto & \left(u + \mathrm{i}\dfrac{\partial \lambda}{\partial v}, \dfrac{\partial \lambda}{\partial u} + \mathrm{i}v\right) \end{array}$$

is an STR immersion if and only if $\det(\mathrm{Hess}\,\lambda) \neq 1$.

4.2.2. *STR immersions of a product of curves*

Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a regular curve of $\mathbb{R}^2$ (I is either $\mathbb{S}^1$ or $\mathbb{R}$). We denote $\gamma(t) = (x(t), y(t))$ and $c(t)$ its curvature. It is easy to check that:

(1) The (regular) curve $\gamma : I \longrightarrow \mathbb{R}^2$ is an STR immersion if and only if $\mathrm{Im}\,\gamma$ lies inside a line.

(2) If $\gamma_1, \ldots, \gamma_n : I \longrightarrow \mathbb{C}$ are n (regular) curves and if

$$\left(\prod_{k=1}^{n} \frac{\partial x_k}{\partial t_k}\right) + \left(\prod_{k=1}^{n} \frac{\partial y_k}{\partial t_k}\right) \neq 0,$$

then the immersion j defined by

$$\begin{aligned} j : I \times \cdots \times I &\longrightarrow \mathbb{C}^n \\ (t_1, \ldots, t_n) &\longmapsto (x_1 + \mathrm{e}^{\mathrm{i}\pi/n} y_n, x_2 + \mathrm{e}^{\mathrm{i}\pi/n} y_1, \ldots, x_n + \mathrm{e}^{\mathrm{i}\pi/n} y_{n-1}) \end{aligned}$$

is an STR. It is an embedding if and only if each $\gamma_k = x_k + i y_k$ is an embedding.

(3) Let

$$\begin{array}{cccc} \delta_k : & I_{2k-1} \times I_{2k} & \longrightarrow & \mathbb{C}^2 \\ & (t_{2k-1}, t_{2k}) & \longmapsto & (x_{2k-1} + i y_{2k}, y_{2k-1} + i x_{2k}). \end{array}$$

If

$$\sum_{k=1}^{p}\left(\frac{\partial x_{2k}}{\partial t_{2k}}\frac{\partial x_{2k-1}}{\partial t_{2k-1}}-\frac{\partial y_{2k}}{\partial t_{2k}}\frac{\partial y_{2k-1}}{\partial t_{2k-1}}\right)\neq 0,$$

then the product immersion $j=\delta_1\times\cdots\times\delta_p$ is an STR. It is an embedding if and only if each $\gamma_k=x_k+iy_k$ is an embedding.

4.2.3. STR immersions of $\mathbb{T}^{n-1}\times\mathbb{R}$

Let $f_1,\ldots,f_n:\mathbb{R}\longrightarrow\mathbb{R}$ be n injective maps such that the derivative of the product $f_1\cdots f_n$ is never zero, then

$$\begin{aligned} j: T^{n-1}\times\mathbb{R} &\longrightarrow \mathbb{C}^n\\ (\theta_1,\ldots,\theta_{n-1},r) &\longmapsto (\mathrm{e}^{\mathrm{i}\theta_1}f_1(r),\ldots,\mathrm{e}^{\mathrm{i}\theta_{n-1}}f_{n-1}(r),\mathrm{e}^{-\mathrm{i}(\theta_1+\cdots+\theta_{n-1})}f_n(r))\end{aligned}$$

is an STR embedding. This ensues directly from the computation of the Jacobian $\mathrm{Jac}_{\mathbb{C}}(\mathrm{d}j^{\mathbb{C}})=(f_1\cdots f_n)'$.

4.2.4. STR immersions of $\mathbb{S}^{2p-1}\times\mathbb{R}$

We obtain STR immersions of $\mathbb{S}^{2p-1}\times\mathbb{R}$ as a direct application of the following observation.

Observation. Let Ω be an open set of $\mathbb{R}^{2p}$ and let $F=(F_1,\ldots,F_{2p}):\Omega\to\mathbb{R}^{2p}$ be a C^∞ function such that $\partial F_i/\partial x_j=\partial F_j/\partial x_i$ for every couple (i,j). Then the following map is special:

$$\begin{aligned}\tilde F:\quad \Omega &\longrightarrow \mathbb{C}^{2p}\\ (x_1,\ldots,x_{2p}) &\longmapsto (x_1+\mathrm{i}F_2,F_1+\mathrm{i}x_2,\ldots,x_{2p-1}+\mathrm{i}F_{2p},F_{2p-1}+\mathrm{i}x_{2p}).\end{aligned}$$

Indeed, after some permutations of rows and some multiplications of columns by i, the Jacobian matrix of $\tilde F$ becomes Hermitian. Thus $\mathrm{Jac}_{\mathbb{C}}(\mathrm{d}\tilde F)=(-\mathrm{i})^p\rho$ for some $\rho\in\mathbb{R}$.

Application. We take $\Omega=\mathbb{R}^{2p}\setminus\{0\}$ which is topologically $\mathbb{S}^{2p-1}\times\mathbb{R}$. Let $F:\mathbb{R}^{2p}\setminus\{0\}\longrightarrow\mathbb{R}^{2p}$ be given by $F(x)=\rho(r)x/r$ where $r=|x|$ and $\rho:\mathbb{R}_*^+\longrightarrow\mathbb{R}_*^+$. Then

$$\frac{\partial F_i}{\partial x_j}=\frac{\rho(r)}{r}\delta_{ij}+\left(\frac{\rho}{r}\right)'\frac{x_ix_j}{r}.$$

Thus $\partial F_i/\partial x_j=\partial F_j/\partial x_i$ and $\tilde F:\mathbb{R}^{2p}\setminus\{0\}\longrightarrow\mathbb{C}^{2p}$ is special. The computation of $\mathrm{Jac}_{\mathbb{C}}(\mathrm{d}\tilde F)$ gives

$$\mathrm{Jac}_{\mathbb{C}}(\mathrm{d}\tilde F)=(-\mathrm{i})^{2p}\frac{(\rho^2-r^2)^{p-1}(r-\rho\rho')}{r^{2p-1}}.$$

Any function ρ such that $\mathrm{Jac}_{\mathbb{C}}(\mathrm{d}\tilde F)\neq 0$ gives an STR immersion of $\mathbb{S}^{2p-1}\times\mathbb{R}$ in $\mathbb{C}^{2p}$. For instance $\rho(r)=r+1/r$ is convenient.

Remark. The immersion $\Phi : \Omega \longrightarrow \mathbb{C}^n$ defined by $\Phi(x) = x + \mathrm{i}F(x)$ is Lagrangian. If $\phi(r)$ is a primitive of $\rho(r)$ then $\mathrm{d}\phi = \rho(r)\,\mathrm{d}r = \rho(r)x/r = F(x)$ and the image set $\Phi(\Omega)$ can be seen as the graph of $\mathrm{d}\phi$. This shows that the order of terms x_i and F_i in the definition of $\tilde{F}$ is crucial.

5. Stability of critical immersions for the J-volume

We assume again M^n orientable, compact, eventually with boundary. We denote by $\tilde{M}^{2n}$ a Kähler manifold.

Definition. Let $j : M^n \longrightarrow \tilde{M}^{2n}$ be a totally real immersion which is critical for the J-volume. The immersion j is said to be *stable for the J-volume* if, for every deformation Φ leaving the boundary of M^n fixed, one has $v''(0) \geq 0$.

Stability problems often come down to a long computation of second derivative. Nevertheless, when $\tilde{M}^{2n} = \mathbb{C}^n$, a straightfoward argument will allow us to avoid such a computation. This is the reason why we have divided the stability study in two parts. In the first part, a slight generalization of the notion of calibration leads us immediately to the result. In the second part, this notion no longer applies and we are compelled to do the computation.

5.1. *First case: $\tilde{M}^{2n} = \mathbb{C}^n$*

5.1.1. *Recall: Calibrated manifolds* [10]

A *calibrated manifold* is a Riemannian manifold $(\tilde{M}^{n+p}, \tilde{g})$ provided with a closed p-form φ such that, for every oriented p-plane Π: $\varphi(\Pi) \leq \mathrm{vol}(\Pi)$. A *$\varphi$-submanifold* is an oriented submanifold M^p of $\tilde{M}^{n+p}$ such that $\varphi_{|M^p} = \mathrm{vol}_{|M^p}$. The notion of φ-submanifold is of central importance in the study of critical submanifolds of the volume functional. If M^p is a compact φ-submanifold of $\tilde{M}^{n+p}$ then M^p minimizes the volume homologically, i.e. for every submanifold N^p such that $\partial N^p = \partial M^p$ and $[M^p - N^p] = 0$ in $H_p(\tilde{M}^{n+p})$, one has $\mathrm{Vol}(M^p) \leq \mathrm{Vol}(N^p)$.

5.1.2. *Notion of J-calibration*

We give here a generalization of the notion of calibration.

Definition.
(1) A *J-calibration* of $\tilde{M}^{2n}$ is a closed n-form φ of $\tilde{M}^{2n}$ such that for every oriented n-plane Π, one has $\varphi(\Pi) \leq \rho_J \,\mathrm{vol}(\Pi)$.
(2) A *φ-submanifold respected to the J-volume* is an oriented submanifold M^n of $\tilde{M}^{2n}$ such that $\varphi_{|M^n} = \rho_J \,\mathrm{vol}_{|M^n}$.

Theorem 4. *Let M^n be a compact submanifold of $\tilde{M}^{2n}$. If M^n is a φ-submanifold respected to the J-volume then M^n minimizes the J-volume homologically, i.e. $V_J(M^n) \leq V_J(N^n)$ for every submanifold N^n such that $\partial N^n = \partial M^n$ and $[M^n - N^n] = 0$ in $H_n(\tilde{M}^{2n})$.*

Proof. It is just a reformulation of the proof of Harvey and Lawson [10]. One has

$$V_J(M^n) = \int_{M^n} \varphi = \int_{N^n} \varphi \leq V_J(N^n). \qquad \square$$

5.1.3. A J-calibration of $\mathbb{C}^n$

Let φ be the closed n-form of $\mathbb{C}^n$ defined by

$$\varphi = \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\beta_0}\, \mathrm{d}z) = \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\beta_0}\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n).$$

Let Π be an n-plane of $\mathbb{C}^n$, $(\epsilon_1, \ldots, \epsilon_n)$ a (direct) orthonormal frame of $\mathrm{Re}(\mathbb{C}^n)$ and $(e_1, \ldots, e_n)$ a (direct) orthonormal frame of Π. There exists $z = r\mathrm{e}^{\mathrm{i}\beta}$ such that $e_1 \wedge \cdots \wedge e_n = r\mathrm{e}^{\mathrm{i}\beta}\epsilon_1 \wedge \cdots \wedge \epsilon_n$. So $\varphi(\Pi) = r\cos(\beta - \beta_0)$. On the other hand $\rho_J \,\mathrm{vol}(\Pi) = r$. Thus,

$$\varphi(\Pi) \leq \rho_J \,\mathrm{vol}(\Pi).$$

Therefore φ is a J-calibration. Equality holds if and only if $\beta = \beta_0$ and φ-submanifolds respected to the J-volume are STR submanifolds. We have proved the following result.

Theorem 5. *Every totally real submanifold M^n of $\mathbb{C}^n$ critical for the J-volume, minimizes the J-volume homologically, i.e. $V_J(M^n) \leq V_J(N^n)$ for every submanifold N^n such that $\partial N^n = \partial M^n$ and $[M^n - N^n] = 0$ in $H_n(\tilde{M}^{2n})$.*

Remark. If M^n is an STR submanifold and N^n a submanifold such that $[M^n - N^n] = 0$, there is no reason for N^n to be totally real. However, $\int_{N^n} \varphi$ is well-defined.

5.2. Second case: $\tilde{M}^{2n}$ is any Kähler manifold

The preceding argument does not apply because we no longer have a 'good' calibration. A (long) computation of second derivative lead us to:

Proposition 3. *Let $\Phi : M^n \times]-\epsilon, \epsilon[\longrightarrow \tilde{M}^{2n}$ be a deformation of M^n leaving the boundary of M^n fixed and such that $\partial\Phi/\partial t = \xi = JX \in JTM^n$. Assume that the J-volume of M^n is finite and that M^n is critical for the J-volume ($v'(0) = 0$). Then*

$$v''(0) = \int_{M^n} \left\{ \frac{1}{\rho_J^2}(\mathrm{div}\, \rho_J X)^2 + 3[\mathrm{d}\ln \rho_J(X)]^2 - \mathrm{Ricci}(X, X) \right\} \rho_J .\, \mathrm{vol}_{M^n}.$$

where $\mathrm{Ricci}(\cdot, \cdot)$ *is the Ricci curvature tensor of $\tilde{M}^{2n}$.*

Theorem 6. *If $\tilde{M}^{2n}$ is a Kähler manifold with non-positive Ricci curvature, then every immersion $j : M^n \longrightarrow \tilde{M}^{2n}$ critical for the J-volume is stable for the J-volume.*

Remark. These results generalize those obtained by B.-Y Chen, P.F. Leung and T. Nagano (see [5]), and by Oh [15] (cf. also [6]).

Proof of Proposition 3. Let $\Phi : M^n \times]-\epsilon, \epsilon[\longrightarrow \tilde{M}^{2n}$ be any deformation of M^n leaving the boundary fixed. We put: $\partial\Phi/\partial t = E = JX + T \in JTM^n \oplus TM^n$. If j is critical, one has

$$v''(0) = \int_{M^n} \frac{G''(0)}{4G(0)^{3/4}} \mathrm{vol}_{M^n}.$$

A first computation gives

$$\mathrm{d}^2G(E, E) = 4G(0)\left\{ \sum_i \theta^i(\tilde{\nabla}_E\tilde{\nabla}_E e_i) + 4\left(\sum_i \theta^i(\tilde{\nabla}_E e_i)\right)^2 \right.$$
$$\left. - \sum_{i,j} \omega_j^i(E)\omega_i^j(E) - \omega_j^{i*}(E)\omega_{i*}^j(E) \right\}. \qquad (*)$$

In the case of a deformation in the direction $E = JX \in JTM^n$, one has

$$\sum_i \theta^i(\tilde{\nabla}_E e_i) = \sum_i \theta^i(\tilde{\nabla}_{e_i} JX) = -\sum_i \theta^{i*}(\tilde{\nabla}_{e_i} X) = -\pi\ \mu(X),$$
$$\sum_i \theta^i(\tilde{\nabla}_E\tilde{\nabla}_E e_i) = -\overline{S}(JX, JX) + \sum_i \theta^i(\tilde{\nabla}_{e_i}\tilde{\nabla}_E JX),$$

where $\overline{S}(Z, Z) = \sum_i \theta^i(\tilde{R}(e_i, Z)Z)$ and $\tilde{R}$ is the curvature tensor of $\tilde{M}^{2n}$. We get

$$\mathrm{d}^2G(JX, JX) = 4G(0)\left\{ -\overline{S}(JX, JX) + \sum_i \theta^i(\tilde{\nabla}_{e_i}\tilde{\nabla}_E JX) + 4\pi^2\ \mu^2(X) \right.$$
$$\left. - \sum_{i,j} \omega_j^i(JX)\omega_i^j(JX) - \omega_j^{i*}(JX)\omega_{i*}^j(JX) \right\}.$$

As $\omega_j^i(JX) = -\omega_j^{i*}(X)$ and $\omega_j^{i*}(JX) = -\omega_j^i(X) + \theta^i([e_j, X])$, we can write

$$\mathrm{d}^2G(JX, JX) = 4G(0)\left\{ -\overline{S}(JX, JX) + \sum_i \theta^i(\tilde{\nabla}_{e_i}\tilde{\nabla}_E JX) + 4\pi^2\ \mu^2(X) \right.$$
$$+ \sum_{i,j} \theta^i([e_j, X])\omega_i^j(X) + \theta^j([e_i, X])\omega_j^i(X)$$
$$+ \theta^i([e_j, X])\theta^j([e_i, X])$$
$$\left. - \sum_{i,j} \omega_j^{i*}(X)\omega_i^{j*}(X) - \omega_j^i(X)\omega_i^j(X) \right\}.$$

Let Ψ be a tangential deformation of M^n leaving the boundary fixed and such that $\partial\Psi/\partial t = X \in TM^n$. Such a deformation does not change the J-volume of the immersion. Thus,

$$\mathrm{d}^2v(JX, JX) = \int_{M^n} \frac{\mathrm{d}^2G(X, X) + \mathrm{d}^2G(JX, JX)}{4G(0)^{3/4}} \mathrm{vol}_{M^n}.$$

Formula (*) yields

$$d^2G(X,X) = 4G(0)\left\{ \sum_i \theta^i(\tilde\nabla_X\tilde\nabla_X e_i) + 4\left(\sum_i \theta^i(\tilde\nabla_X e_i)\right)^2 - \sum_{i,j} \omega_j^i(X)\omega_i^j(X) - \omega_j^{i*}(X)\omega_{i*}^j(X)\right\}.$$

Therefore,

$$\begin{aligned} d^2G(X,X) &+ d^2G(JX,JX) \\ = 4G(0)\Bigg\{ &- \overline{S}(JX,JX) + 4\left(\sum_i \theta^i(\tilde\nabla_X e_i)\right)^2 \\ &+ \sum_i \theta^i(\tilde\nabla_X\tilde\nabla_X e_i) + \sum_i \theta^i(\tilde\nabla_{e_i}\tilde\nabla_E JX) \\ &+ \sum_{i,j} \theta^i([e_j,X])\omega_i^j(X) + \theta^j([e_i,X])\omega_j^i(X) \\ &+ \sum_{i,j} \theta^i([e_j,X])\theta^j([e_i,X]) + 4\pi^2\,\mu^2(X)\Bigg\}. \end{aligned}$$

It is easy to verify that

$$\sum_j \theta^j(\tilde\nabla_X\tilde\nabla_X e_j) = \sum_j (\tilde\nabla_X\tilde\nabla_{e_j}X) + \sum_{i,j}\theta^i([X,e_j])\omega_i^j(X) - X\,\mathrm{div}X$$

$$\sum_j \tilde\nabla_{[e_j,X]}X = \sum_{i,j}\theta^i([e_j,X])\omega_i^j(X) + \sum_{i,j}\theta^i([e_j,X])\theta^j([e_i,X]).$$

Thus,

$$\begin{aligned} d^2G(X,X) &+ d^2G(JX,JX) \\ = 4G(0)\Bigg\{ &- \overline{S}(JX,JX) - \overline{S}(X,X) + 4\left(\sum_i \theta^i(\tilde\nabla_X e_i)\right)^2 + \sum_i \theta^i(\tilde\nabla_{e_i}\tilde\nabla_E JX) \\ &+ \sum_i \theta^i(\tilde\nabla_{e_i}\tilde\nabla_X X) - X\,\mathrm{div}X + 4\pi^2\,\mu^2(X)\Bigg\}. \end{aligned}$$

A computation shows that if $Z = U + JV \in TM^n \oplus JTM^n$ is any vector field, one has

$$\rho_J \sum_i \theta^i(\tilde\nabla_{e_i} Z) = \mathrm{div}(\rho_J U) - \pi\,\rho_J\mu(V).$$

Moreover, for a critical immersion $\mu \equiv 0$, so

$$d^2G(X,X) + d^2G(JX,JX)$$

$$= 4G(0)\Bigg\{ -\overline{S}(JX, JX) - \overline{S}(X, X) - X\,\mathrm{div}X + 4\left(\sum_i \theta^i(\tilde{\nabla}_X e_i)\right)^2 + \mathrm{div}(\pi(\tilde{\nabla}_X X + \tilde{\nabla}_E E))\Bigg\},$$

where π is the projection over TM^n parallel to JTM^n. Since $G(0) = \rho_J^4$, $E_{|\partial M^n} = 0$ and $\sum_i \theta^i(\tilde{\nabla}_X e_i) = d\ln\rho_J(X)$, we get

$$\begin{aligned} &\mathrm{d}^2 v(JX, JX) \\ &\quad = \int_{M^n} \{4(\mathrm{d}\ln\rho_J(X))^2 - X\,\mathrm{div}X - \overline{S}(JX, JX) - \overline{S}(X, X)\}\rho_J\,\mathrm{vol}_{M^n} \\ &\quad = \int_{M^n} \left\{4\frac{(\mathrm{d}\ln\rho_J(X))^2}{\rho_J} - \rho_J X\,\mathrm{div}X\right\}\mathrm{vol}_{M^n} - \int_{M^n} \mathrm{Ricci}(X, X)\rho_J\,\mathrm{vol}_{M^n}. \end{aligned}$$

Expanding $\mathrm{div}(\rho_J\mathrm{div}(X)X)$, we obtain

$$-\rho_J X\,\mathrm{div}\,X = \frac{(\mathrm{div}\,\rho_J X)^2}{\rho_J} - \frac{(\mathrm{d}\rho_J(X))^2}{\rho_J} - \mathrm{div}(\rho_J\,\mathrm{div}(X)X).$$

Using this, we get

$$\begin{aligned} \mathrm{d}^2 v(JX, JX) = &\int_{M^n} \left\{\frac{(\mathrm{div}\rho_J X)^2}{\rho_J} + 3\frac{(\mathrm{d}\rho_J(X))^2}{\rho_J} - \mathrm{div}(\rho_J\,\mathrm{div}\,(X)X)\right\}\mathrm{vol}_{M^n} \\ &- \int_{M^n} \mathrm{Ricci}(X, X)\rho_J\,\mathrm{vol}_{M^n}. \end{aligned}$$

Seeing that $E_{|\partial M^n} = 0$, we finally get

$$\mathrm{d}^2 v(JX, JX) = \int_{M^n} \left\{\frac{1}{\rho_J^2}(\mathrm{div}\rho_J X)^2 + 3[\mathrm{d}\ln\rho_J(X)]^2 - \mathrm{Ricci}(X, X)\right\}\rho_J.\,\mathrm{vol}_{M^n}.$$

This finishes the proof of proposition 3. □

Theorem 6 is an easy consequence of this proposition.

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